Lecture 28

Absolute continuity

<u>Def</u>. If ν signed, μ positive measures on $(X, \mathcal{M})$, then $\nu \ll \mu$ (ν <u>absolutely continuous</u> w.r.t. μ) provided

$$\mu(E) = 0 \Rightarrow \nu(E) = 0. \quad (\Rightarrow E \text{ null for } \nu)$$

<u>Rem</u>. $\nu \ll \mu \Leftrightarrow |\nu| \ll \mu \Leftrightarrow \nu^+, \nu^- \ll \mu$. (HW)

<u>Thm 1</u>. Assume ν finite, signed and μ positive measures. Then

$$\nu \ll \mu \Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \mu(E) < \delta \Rightarrow |\nu|(E) < \varepsilon.$$

<u>Pf</u>. Suffices to consider the special case where ν also positive. "$\Leftarrow$" is obvious. We prove "$\Rightarrow$".

Suppose not. Then $\exists\, \varepsilon > 0$ and $E_k \in \mathcal{M}$ s.t. $\mu(E_k) \le 2^{-k}$ but $\nu(E_k) \ge \varepsilon$.

Let $F_n = \bigcup_{k=n}^{\infty} E_k$ (decreasing), $F = \bigcap_{n=1}^{\infty} F_n$

$\Rightarrow$

$$\begin{cases} \nu(F_n) \ge \nu(E_n) \ge \varepsilon \\ \mu(F_n) \le \sum_{k=n}^{\infty} \mu(E_k) \le 2^{-n+1}. \end{cases}$$

Since $F \subseteq F_n$, $\forall n$, $\Rightarrow \mu(F) \le 2^{-n+1} \xrightarrow[n\to\infty]{} 0$

$\Rightarrow \mu(F) = 0$.

Since ν finite, $\overbrace{\text{cont. from above}}^{F_n \downarrow F} \Rightarrow$

$\nu(F) = \lim_{n\to} \nu(F_n) \ge \varepsilon$.

$\Rightarrow \nu \cancel{\ll} \mu$. Thus, $\nu \ll \mu \Rightarrow$ ε-δ statement.

$\boxtimes$

<u>Lebesgue-Radon-Nikodym Thm</u> Let ν be σ-finite signed, μ σ-finite positive measure on $(X, \mathcal{M})$. Then, $\exists$ unique σ-finite signed measures λ, ρ s.t.

$\nu = \lambda + \rho$ and $\lambda \perp \mu$, $\rho \ll \mu$.

Moreover, there is an extended μ-integrable $f: X \to \mathbb{R}$ (f^+ or f^- in L^1) s.t. $d\rho = f\,d\mu$ (i.e. $\rho(E) = \int_E f\,d\mu$) and f is unique μ-a.e.

<u>Rem.</u> • If $\nu \ll \mu$, then LRN $\Rightarrow$ $d\nu = f\,d\mu$. The (μ-a.e. defined) fcn f is called Radon-Nikodym derivative of ν w.r.t. μ: $f = \frac{d\mu}{d\nu}$.

• Have $\int g\,d\nu = \int g \frac{d\nu}{d\mu} d\mu$.

• "Chain rule": If $\nu \ll \rho$, $\rho \ll \mu \Rightarrow$ $\nu \ll \mu$ and $\frac{d\nu}{d\mu} = \frac{d\nu}{d\rho}\frac{d\rho}{d\mu}$

• See Prop 3.9 in Folland for details.

Pf of LRN: We shall do the special case where ν, μ both finite and positive. Going from finite to σ-finite proceeds along "standard lines", and allowing ν to be signed utilizes the Jordan decomposition $\nu = \nu^+ - \nu^-$. Details are DIY or see Folland.

Consider the family (which contains $f \equiv 0$)

$$\mathcal{F} = \{ f : X \to [0,\infty] : \int_E f\,d\mu \le \nu(E) \text{ for all } E \in \mathcal{M} \}$$

and let $\alpha = \sup_{\mathcal{F}} \int_X f\,d\mu$ $(\le \nu(E) < \infty)$.

We take $f_n \in \mathcal{F}$ s.t. $\int_X f_n\,d\mu \nearrow \alpha$.

Note. If $f_1, f_2 \in \mathcal{F}$, then $g = \max(f_1, f_2)$ is also in $\mathcal{F}$. To see this, let $A = \{x : f_1(x) \le f_2(x)\}$. Thus, $g|_A = f_2$ and $g|_{X \setminus A} = f_1$. For $E \in \mathcal{M}$

$$\int_E g\,d\mu = \int_{E \cap A} f_2\,d\mu + \int_{E \cap (X \setminus A)} f_1\,d\mu \le \longrightarrow$$

$\leq \nu(E\cap A)+\nu(E\cap(X\setminus A))=\nu(E)$.

Form $g_n=\max(f_1,\dots,f_n)\in\mathcal{F}$. Then, $g_n\in L^+$ and $g_n\nearrow$. Define $f(x)=\lim_{n\to\infty} g_n(x)$

$\Rightarrow f:X\to[0,\infty]$ and, by MCT,

$$\int_E f\,d\mu=\lim_{n\to\infty}\int_E g_n\,d\mu\leq\nu(E)\Rightarrow f\in\mathcal{F}.$$

Since $\int_X f_n\,d\mu\leq\int_X g_n\,d\mu \Rightarrow \int f\,d\mu=\alpha$.

Note that $f\in L^+$, $\int f\,d\mu=\alpha<\infty\Rightarrow$ $\mu(f^{-1}(\{\infty\}))=0\Rightarrow$ change f on this set so that $f:X\to[0,\infty)$ w/ $f\in L^1$.

Define ρ by $d\rho=f\,d\mu$ and λ by $d\lambda=d\nu-f\,d\mu\Rightarrow \nu=\lambda+\rho$.

<u>Claim</u>. $\lambda\perp\mu$.

For the pf of Claim, we shall need:

<u>Lemma1</u>. If $\lambda \not\perp \mu$, then $\exists E$ s.t. $\mu(E) > 0$ and $\varepsilon > 0$ s.t. E is positive for $\lambda - \varepsilon\mu$.

<u>Pf</u>. Consider the signed measures $\nu_n = \lambda - \frac{1}{n}\mu$, and let $X = P_n \cup N_n$ be HD for each. Let $P = \bigcup_{n=1}^{\infty} P_n$, $N = \bigcap_{n=1}^{\infty} N_n = X \setminus P$.

N is neg. for all $\nu_n \Rightarrow \lambda(N) \le \frac{1}{n}\mu(N)$ for all $N \Rightarrow \lambda(N) = 0$. If $\mu(P) = 0$, then $\lambda \perp \mu$. If not, then we must have $0 < \mu(P) \le \sum_{n=1}^{\infty} \mu(P_n) \Rightarrow$ $\mu(P_n) > 0$ for some n. Then, taking $\varepsilon = \frac{1}{n}$, $E = P_n$ yields the other alternative. $\boxtimes$

To prove claim, we note that if $\lambda \not\perp \mu$, then $\exists E, \varepsilon > 0$ as in Lemma1. Then, $\forall E' \in \mathcal{M}$, $E' \subseteq E$

$0 \le \lambda(E') - \varepsilon\mu(E') = \int_{E'} d\lambda - \int_{E'} \varepsilon \, d\mu = \longrightarrow$

$$= \int_{E'} d\nu - \int_{E'} (f+\varepsilon)\, d\mu \Rightarrow \int_{E'} (f+\varepsilon)\, d\mu \leq \nu(E').$$

Consider $g = f + \varepsilon \chi_E$. Let $F \in \mathcal{M} \Rightarrow$

$$\int_F g\, d\mu = \int_{F\cap E} (f+\varepsilon)\, d\mu + \int_{F\cap E^c} f\, d\mu \underset{\text{By above } + f\in\mathcal{F}}{\leq} \nu(F\cap E) +$$

$$+ \nu(F\cap E^c) = \nu(F) \Rightarrow g \in \mathcal{F}.$$

But then $\int_X g\, d\mu = \int_X f\, d\mu + \varepsilon\mu(E) = \alpha + \varepsilon\mu(E)$

Since $\alpha = \sup_{g\in\mathcal{F}} \int_X g\, d\mu$, we have a contradiction

$\Rightarrow \lambda \perp \mu$, which is the desired LRN-decomp.

The uniqueness is easy. If $\nu = \lambda + \rho = \lambda' + \rho'$, then $d(\lambda - \lambda') = (f - f')\, d\mu$. Since $\lambda \perp \mu$, $\lambda' \perp \mu$, it is easy to see that $\lambda - \lambda' \perp \mu$.

But then $d(\lambda - \lambda') \perp (f - f')\, d\mu$. $\Rightarrow \lambda - \lambda' = 0$ and $(f - f')\, d\mu = 0 \Rightarrow f - f' = 0$ μ-a.e.

This completes the pf. ∎

Concluding Remarks.

Consider Lebesgue-Stieltjes measure μ_F on $\mathbb{R}$. Since μ_F is defined by

$$\mu_F\Big(\underbrace{\bigcup_{k=1}^{\infty}(a_k, b_k]}_{\text{disjoint}}\Big) = \sum_{k=1}^{\infty}\big(F(b_k)-F(a_k)\big)$$

we see, by Thm, that abs. cont. of μ_F wrt to $m \Rightarrow \forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$\sum_{k=1}^{\infty}|b_k-a_k| < \varepsilon \Rightarrow \sum_{k=1}^{\infty}\underbrace{\big(F(b_k)-F(a_k)\big)}_{=|F(b_k)-F(a_k)|} < \infty$$

New notion of continuity! Call this <u>absolute continuity</u> of F. If we let F' be Radon-Nikodym derivative of F, then we have "Fundamental Thm of Calculus" for abs. cont. functions F

$$\int_a^b F'(x)\,dx = F(b)-F(a).$$

<u>Note</u>:
$C^1 \Rightarrow$ differentiable $\Rightarrow$ abs. cont. $\Rightarrow$ uniform cont. $\Rightarrow$ cont.